

\mathbb{T}^ω -representations of compact sets

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We explore representing the compact subsets of a given represented space by infinite sequences over Plotkin's \mathbb{T} . We show that compact Hausdorff spaces with a proper dyadic subbase admit representations of their compact subsets in such a way that compact sets are essentially underspecified points. We can even ensure that a name of an n -element compact set contains at most n occurrences of \perp .

1 Introduction

In TTE [16], the fundamental computability notion is introduced on either Cantor space $\{0, 1\}^\omega$ or Baire space \mathbb{N}^ω , and then lifted to other spaces of interest via representations. It is well-known that the choice of $\{0, 1\}^\omega$ or \mathbb{N}^ω is inconsequential for the resulting theory, and authors typically choose whatever space works better for a specific purpose. In principle, other spaces can be used as the fundament, too, provided that they have a sufficiently substantial computability theory defined on them. Using the space of regular word functions has been advocated by KAWAMURA and COOK with computational complexity as the motivation [3]. If one is primarily interested in Quasi-Polish spaces [1], then the Scott domain $\mathcal{P}(\omega)$ makes sense as the foundational space, with a computability notion derived from enumeration reducibility (cf. [4]).

Here we consider \mathbb{T}^ω as a fundamental space for computation, the space of infinite sequences over Plotkin's \mathbb{T} . Plotkin's \mathbb{T} is the three point space $\{0, 1, \perp\}$ with the topology generated by $\{\{0\}, \{1\}\}$. Thus, \perp plays the role of *not yet determined*, whereas the values 0 and 1, once attained, will remain unchanged. The use of \mathbb{T}^ω (together with IM2 machines) as the basis for a theory of computability has been investigated by the second author in a number of papers [11, 14, 10]. An interesting result is that a computable metric space \mathbf{X} admits an injective representation $\delta : \subseteq \mathbb{T}^\omega \rightarrow \mathbf{X}$ such that each $p \in \text{dom}(\delta)$ has at most n occurrences of \perp iff the dimension of \mathbf{X} is at most n .

In the present paper we consider \mathbb{T}^ω -representations of the space $\mathcal{K}(\mathbf{X})$ of compact subsets of some space \mathbf{X} represented over $\{0, 1\}^\omega$. We are particularly interested in matching representations in the following sense:

Definition 1. Consider a representation $\delta : \subseteq \{0, 1\}^\omega \rightarrow \mathbf{X}$ and a \mathbb{T}^ω -representation $\psi : \subseteq \mathbb{T}^\omega \rightarrow \mathcal{K}(\mathbf{X})$. We say that they *match*, iff:

$$\begin{aligned} x \in A &\Leftrightarrow \forall p \in \psi^{-1}(A) \exists q \in \{0, 1\}^\omega \quad q \succeq p \wedge \delta(q) = x \\ &\Leftrightarrow \exists p \in \psi^{-1}(A) \exists q \in \{0, 1\}^\omega \quad q \succeq p \wedge \delta(q) = x \end{aligned}$$

Here \preceq denotes the specialization relation on $\mathbb{T}^\omega \supset \{0, 1\}^\omega$.

A pair of matching representations essentially means that we can consider compact subsets as underspecified points. This seems like a counterpart to the identification of points in admissible spaces as being equivalent to compact singletons [8, 7]. Ideally, we want even more: We would like the names of the compact subsets to be as little underspecified as possible, i.e. containing as few occurrences of \perp as possible. This in turn will be linked to having little redundancy in how many different δ -names for the same point are obtainable from a ψ -name of a compact set containing the point. We will provide constructions of such matching representations based on the concept of a proper dyadic subbase.

Our constructions were inspired by, and yield a uniform version of, a construction employed to prove [5, Proposition 1.9].

2 Notation and Fundamentals

Background on represented spaces

We briefly recall some fundamental concepts on represented spaces following [7], to which the reader shall also be referred for a more extensive treatment. A *represented space* is a pair $\mathbf{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X : \subseteq \{0, 1\}^\omega \rightarrow X$. A (multivalued) function between represented spaces is a (multivalued) function between the underlying sets. For $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ and $F : \subseteq \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$, we call F a *realizer* of f (notation $F \vdash f$), iff $\delta_Y(F(p)) = f(\delta_X(p))$ for all $p \in \text{dom}(f\delta_X)$, i.e. if the following diagram commutes:

$$\begin{array}{ccc} \{0, 1\}^\omega & \xrightarrow{F} & \{0, 1\}^\omega \\ \downarrow \delta_{\mathbf{X}} & & \downarrow \delta_{\mathbf{Y}} \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

A map between represented spaces is called *computable* (continuous), iff it has a computable (continuous) realizer. A priori, the notion of a continuous map between represented spaces and a continuous map between topological spaces are distinct. However, for the admissible spaces (as defined by SCHRÖDER [9, 8]) the notions coincide. This in particular includes the computable metric spaces.

Cantor space itself is considered a represented space, with $\text{id} : \{0, 1\}^\omega \rightarrow \{0, 1\}^\omega$ serving as the representation. Other specific relevant represented spaces are \mathbb{N} represented by $\delta_{\mathbb{N}}(0^n 1^\omega) = n$ and Sierpiński space \mathbb{S} with underlying set $\{\perp, \top\}$ and representation $\delta_{\mathbb{S}}$ defined by $\delta_{\mathbb{S}}(0^\omega) = \perp$ and $\delta_{\mathbb{S}}(p) = \top$ for $p \neq 0^\omega$.

Courtesy of the UTM-theorem (in the form proven in [15]), there is a natural representation of the space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of continuous functions between two given represented spaces \mathbf{X}, \mathbf{Y} . Then representation is rendering all the expected operations computable, in particular composition and evaluation. We immediately obtain a representation of any \mathbf{X}^ω in form of $\mathcal{C}(\mathbb{N}, \mathbf{X})$. We can also derive the space $\mathcal{O}(\mathbf{X})$ of open subsets of \mathbf{X} by identifying a set $U \subseteq \mathbf{X}$ with its characteristic function $\mathcal{C}(\mathbf{X}, \mathbb{S}) \ni \chi_U : \mathbf{X} \rightarrow \mathbb{S}$ mapping $x \in U$ to \top and $x \notin U$ to \perp . The open subsets are the final topology along the representation, and again, the expected operations are computable.

The space $\mathcal{A}(\mathbf{X})$ of closed subsets by considering the characteristic function of its complement. In other words, we define $\mathcal{A}(\mathbf{X})$ in such a way that $^C : \mathcal{O}(\mathbf{X}) \rightarrow \mathcal{A}(\mathbf{X})$ and $^C : \mathcal{A}(\mathbf{X}) \rightarrow$

$\mathcal{O}(\mathbf{X})$ became computable. We further introduce the space $\mathcal{K}(\mathbf{X})$ of compact subsets by representing $A \subseteq \mathbf{X}$ via $\{U \in \mathcal{O}(\mathbf{X}) \mid A \subseteq U\} \in \mathcal{O}(\mathcal{O}(\mathbf{X}))$.

A represented space \mathbf{X} is called computably compact, if $\text{isEmpty} : \mathcal{A}(\mathbf{X}) \rightarrow \mathbb{S}$ is computable. It is called computably Hausdorff, iff $\neq : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{S}$ is computable. A space is computably compact and computably Hausdorff iff both $\text{id} : \mathcal{A}(\mathbf{X}) \rightarrow \mathcal{K}(\mathbf{X})$ and $\text{id} : \mathcal{K}(\mathbf{X}) \rightarrow \mathcal{A}(\mathbf{X})$ are well-defined and computable. As we will be working with (computable) compact Hausdorff spaces, we can freely alternate between treating sets represented as closed or as compact sets in the following.

Introducing \mathbb{T}^ω -represented spaces

We can consider \mathbb{T} as a represented space (over $\{0, 1\}^\omega$) via the representation $\delta_{\mathbb{T}} : \{0, 1\}^\omega \rightarrow \mathbb{T}$ defined by $\delta_{\mathbb{T}}(0^\omega) = \perp$, $\delta_{\mathbb{T}}(p) = 0$ iff $\min\{n \in \mathbb{N} \mid p(n) = 1\}$ is even and $\delta_{\mathbb{T}}(p) = 1$ iff $\min\{n \in \mathbb{N} \mid p(n) = 1\}$ is odd. From this representation we derive a representation $\delta_{\mathbb{T}^\omega}$ of \mathbb{T}^ω in the usual way; and thus have a notion of computability of (multivalued) functions on \mathbb{T}^ω available. We could alternatively define computability on \mathbb{T}^ω directly via IM2 machines, but will not do so here for sake of simplicity.

A \mathbb{T}^ω -representation ψ of some set X is just a partial surjection $\psi : \subseteq \mathbb{T}^\omega \rightarrow X$, and a \mathbb{T}^ω -represented space is a set equipped with a \mathbb{T}^ω -representation of it. As $\{0, 1\}^\omega \subset \mathbb{T}^\omega$, we can consider every (ordinary) representation as a special case of a \mathbb{T}^ω -representation. Conversely, every \mathbb{T}^ω -representation ψ induces an ordinary representation $\psi \circ \delta_{\mathbb{T}^\omega}$.

Let \mathbf{X}, \mathbf{Y} be \mathbb{T}^ω -represented spaces. We call a multivalued function $F : \subseteq \mathbb{T}^\omega \rightrightarrows \mathbb{T}^\omega$ a \mathbb{T}^ω -realizer of $f : \subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ iff $\emptyset \neq \delta_Y(F(p)) \subseteq f(\delta_X(p))$ for all $p \in \text{dom}(f\delta_X)$. Unlike the situation for ordinary representations, we also need multivalued realizers here. The reason is that not every computable multivalued function $F : \subseteq \mathbb{T}^\omega \rightrightarrows \mathbb{T}^\omega$ has a computable choice function¹. Again, we call a (multivalued) function between \mathbb{T}^ω -represented spaces computable (continuous), iff it has a computable (continuous) \mathbb{T}^ω -realizer. The following is then straight-forward:

- Proposition 2.** 1. Let \mathbf{X} and \mathbf{Y} be represented spaces. A multivalued function $f : \subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is computable (continuous) as a function between represented spaces iff it is computable (continuous) as a function between \mathbb{T}^ω -represented spaces.
2. Let \mathbf{X} and \mathbf{Y} be \mathbb{T}^ω -represented spaces, and $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ the induced represented spaces. Then $f : \subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is computable (continuous) iff $f : \subseteq \overline{\mathbf{X}} \rightrightarrows \overline{\mathbf{Y}}$ is.

Notation

By \mathbb{T}^* we denote the space of finite sequences over \mathbb{T} with known length. We consider $\iota : \mathbb{T}^* \rightarrow \mathbb{T}^\omega$ mapping $w \in \mathbb{T}^*$ to $w\perp^\omega$ as a standard computable map, but point out that the partial inverse of ι is not computable. For $w \in \mathbb{T}^*$, we let $|w| \in \mathbb{N}$ denote its length.

We call the number of digits (i.e., 0 or 1) in a (finite or infinite) bottomed sequence its level. We denote by $\mathbb{T}_{\langle n \rangle}^*$ the set of level- n finite bottomed sequences. More generally, for a subset A of \mathbb{T}^* , we denote by $A_{\langle n \rangle}$ the set of level- n finite bottomed sequences of A . $101, 1\perp 1, 1, \epsilon$ belong to $\mathbb{T}_{\langle 3 \rangle}^*, \mathbb{T}_{\langle 2 \rangle}^*, \mathbb{T}_{\langle 1 \rangle}^*, \mathbb{T}_{\langle 0 \rangle}^*$, respectively. Likewise, $\mathbb{T}_{\langle n \rangle}^\omega$ denotes the set of infinite sequences

¹For example, consider $G : \mathbb{T}^\omega \rightrightarrows \mathbb{T}^\omega$ defined by $G(p) = \{0^\omega\}$ iff $p(0) = \perp$ and $G(p) = \{0, 1\}^\omega \setminus \{0^\omega\}$ iff $p(0) \neq \perp$.

with n digits. We further write $\mathbb{T}_{\langle \leq n \rangle}^* := \bigcup_{i \leq n} \mathbb{T}_{\langle i \rangle}^*$, $\mathbb{T}_{\langle * \rangle}^* := \bigcup_{i \in \mathbb{N}} \mathbb{T}_{\langle i \rangle}^*$, $\mathbb{T}_{\langle \leq n \rangle}^\omega := \bigcup_{i \leq n} \mathbb{T}_{\langle i \rangle}^\omega$ and $\mathbb{T}_{\langle * \rangle}^\omega := \bigcup_{i \in \mathbb{N}} \mathbb{T}_{\langle i \rangle}^\omega$.

For $p, q \in \mathbb{T}^\omega$ we write $p \preceq q$ if $\forall n \in \mathbb{N} (p(n) \neq \perp \Rightarrow p(n) = q(n))$. We write $\text{dom}(p) = \{n \in \mathbb{N} \mid p(n) \neq \perp\}$. By $p \uparrow q$ we denote that $\forall n \in \text{dom}(p) \cap \text{dom}(q) p(n) = q(n)$. We extend these notions to \mathbb{T}^* along ι . Note that \preceq is not a quasiorder on \mathbb{T}^* , as e.g. $0 \preceq 0\perp$ and $0\perp \preceq 0$. By excluding finite sequences ending in \perp we obtain canonic representatives of each \preceq -equivalence class, and more over, turn ι into an embedding.

For $p \in \mathbf{X}^\omega$ or $p \in \mathbf{X}^*$, we denote by $p_{\leq n} \in \mathbf{X}^n$ its restriction to first n components. In the case $p \in \mathbf{X}^*$, we assume that $n \leq |p|$ in this.

By $\mathbf{3}$ we denote the set $\{0, 1, \perp\}$ equipped with the representation $\delta_{\mathbf{3}}(10^\omega) = 0$, $\delta_{\mathbf{3}}(110^\omega) = 1$ and $\delta_{\mathbf{3}}(1110^\omega) = \perp$. Clearly $\text{id} : \mathbf{3} \rightarrow \mathbb{T}$ is computable, but $\text{id} : \mathbb{T} \rightarrow \mathbf{3}$ is not. In the following, we will suppress both $\iota : \mathbb{T}^* \rightarrow \mathbb{T}^\omega$ and $\text{id} : \mathbf{3} \rightarrow \mathbb{T}$ and their combinations in the notation. For example, if we have some function $f : \mathbb{T}^\omega \rightarrow \mathbf{X}$, we might speak of the function $f : \mathbf{3}^* \rightarrow \mathbf{X}$ obtained by precomposing with these computable functions without further notice.

3 \mathbb{T}^ω -representations of pruned trees

For $w, v \in \{0, 1\}^*$, let $w \sqsubseteq v$ express that w is a prefix of v . We recall that a (binary) tree is a set $T \subseteq \{0, 1\}^*$ such that $v \in T \wedge w \sqsubseteq v$ implies $w \in T$. The elements of a tree are called vertices. By using some standard bijection $\nu : \mathbb{N} \rightarrow \{0, 1\}^*$, we can then represent a tree T by its characteristic function $\chi_T \in \{0, 1\}^\omega$. We shall denote the space of binary trees by \mathcal{T} .

Some $p \in \{0, 1\}^\omega$ is called an infinite path through a tree T if $\forall n \in \mathbb{N} p_{\leq n} \in T$. The set of infinite paths through T is denoted by $[T]$. It is well-known that the closed subsets of $\{0, 1\}^\omega$ arise as $[T]$ for some tree in a uniform way. In other words $T \mapsto [T] : \mathcal{T} \rightarrow \mathcal{A}(\{0, 1\}^\omega)$ is computable and has a computable multivalued inverse.

A tree is pruned, if $w \in T$ implies $\exists v \in T w \sqsubset v$. By induction, in a pruned tree every vertex is the prefix of some infinite path through it. Moreover, for any tree T there is a unique pruned tree T_p such that $[T] = [T_p]$ – however, T_p is not computable from T , with the Kleene tree being the canonic counterexample.

Definition 3. We define the \mathbb{T}^ω -represented space PT by letting the underlying set be the pruned binary trees, and the \mathbb{T}^ω representation $\delta_{\text{PT}} : \mathbb{T}^\omega \rightarrow \text{PT}$ be defined as follows:

$$\begin{aligned} \delta_{\text{PT}}(p) = T \Leftrightarrow & [(\nu(n) \in T \wedge p(n+1) \neq 1 \rightarrow \nu(n)0 \in T) \\ & \wedge (\nu(n) \in T \wedge p(n+1) \neq 0 \rightarrow \nu(n)1 \in T) \\ & \wedge (p(0) = \perp \Leftrightarrow T \neq \emptyset)] \end{aligned}$$

This means that we use the first symbol in a δ_{PT} -name for a tree to indicate whether the tree is empty or not, with \perp representing non-emptiness. If the tree is non-empty, then clearly $\varepsilon \in T$. Then for any vertex w of the tree, the value of $p(\nu^{-1}(w) + 1)$ indicates whether the left child, the right child or both are part of the tree. This represents precisely the pruned trees, as there the fourth case of *neither* does not apply.

Theorem 4. The map $\text{Prune} : \mathcal{T} \rightarrow \text{PT}$ is computable and has a computable multivalued inverse.

Proof. We compute the pruned tree $T' \in \text{PT}$ from $T \in \mathcal{T}$, we read through T layer by layer (i.e. consider all $w \in T$ with $|w| = n$ at the same. While doing so, we construct a δ_{PT} -name q of T' . We can assume that initially, $q = \perp^\omega$, and then change entries in q to 0 or 1 as required.

If at any stage of the computation we find that $q(\nu(w) + 1) = \perp$ and there is some $k \in \mathbb{N}$ such that for all $v \sqsupseteq w0$ with $|v| = k$ we learn that $v \notin T$, then we set $q(\nu^{-1}(w) + 1) := 1$. If $q(\nu^{-1}(w) + 1) = \perp$ and there is some $k \in \mathbb{N}$ such that for all $v \sqsupseteq w1$ with $|v| = k$ we learn that $v \notin T$, then we set $q(\nu^{-1}(w) + 1) := 0$. If we ever find some $k \in \mathbb{N}$ such that $v \notin T$ for all v with $|v| = k$, then we set $q(0) = 0$. Using compactness, it is straight-forward to verify that this yields a valid δ_{PT} -name for T' . Note that moreover, every entry in the resulting name q which is not specified by the definition of δ_{PT} will be either 0 or 1, but never \perp .

Now let us consider how to compute the multivalued inverse of Prune. We start with some $\delta_{\mathbb{T}^\omega}$ -name q of a δ_{PT} -name of some pruned tree T . For $w \in \{0, 1\}^*$ and $n \in \mathbb{N}$, let $t_{w,n} := \delta_{\text{PT}}(\delta_{\mathbb{T}^\omega}(q_{\leq n} 0^\omega))(\nu^{-1}(w) + 1)$. Let $s_n := \delta_{\text{PT}}(\delta_{\mathbb{T}^\omega}(q_{\leq n} 0^\omega))(0)$. Note that given q , n , w we can compute $t_{w,n} \in \mathbf{3}$ and $s_n \in \mathbf{3}$.

We define a tree $T' \in \mathcal{T}$ by setting $v \in T'$ iff $s_{|v|} = \perp \wedge \forall k < |v| \ t_{v_{\leq k}, |v|} = \perp \vee t_{v_{\leq k}, |v|} = v(k+1)$. This is a tree by monotonicity of the condition (which is in part derived from the monotonicity of s_n and $t_{w,n}$ in n). As $t_{w,n} \in \mathbf{3}$ is available, the condition is decidable, and thus the tree is known as an element of \mathcal{T} . It is straight-forward to verify that $[T'] = [T]$. \square

Corollary 5. There is a surjection $t_{\{0,1\}^\omega} : \mathbb{T}^\omega \rightarrow \mathcal{A}(\{0,1\}^\omega)$ and a multivalued map $s_{\{0,1\}^\omega} : \mathcal{A}(\{0,1\}^\omega) \rightrightarrows \mathbb{T}^\omega$ such that

1. $s_{\{0,1\}^\omega}$ and $t_{\{0,1\}^\omega}$ are computable.
2. $t_{\{0,1\}^\omega} \circ s_{\{0,1\}^\omega} = \text{id}_{\mathcal{A}(\{0,1\}^\omega)}$.
3. If $A \in \mathcal{A}(\{0,1\}^\omega)$ is a finite set, then the cardinality of A is equal to the number of \perp in $s_{\{0,1\}^\omega}(A)$.

Proof. We obtain $t_{\{0,1\}^\omega}$ as $(T \mapsto [T]) \circ (\text{Prune}^{-1}) \circ \delta_{\text{PT}}$, and then s as its multivalued inverse. Property 3 follows from the observation in the proof of Theorem 4 that in names resulting from Prune all non-specified entries are not \perp . In a non-empty set, the first entry is \perp . Every further \perp corresponds to a branching in the tree, which in turn increases the number of points by 1. \square

Observation 6. Every continuous $f : \mathbb{T}^\omega \rightarrow \mathcal{A}(\mathbf{X})$ is monotonic, i.e. $p \preceq q \Rightarrow f(p) \supseteq f(q)$.

Proposition 7. Let \mathbf{X} be uncountable and T_1 . Then there is no computable injection $s : \subseteq \mathcal{A}(\mathbf{X}) \rightarrow \{p \in \mathbb{T}^\omega \mid |\mathbb{N} \setminus \text{dom}(p)| < \infty\}$ with $\text{dom}(s) = \{A \in \mathcal{A}(\mathbf{X}) \mid |A| \in \{1, 2\}\}$.

Proof. For cardinality reasons, there have to be sets $A, B \in \mathcal{A}(\mathbf{X})$ with $|A| = |B| = 1$ such that $s(A)$ and $s(B)$ differ at infinitely many positions. Now consider $s(A \cup B)$: Continuity of s implies that arbitrarily late, $s(A \cup B)$ can change to either $s(A)$ or $s(B)$, i.e. $s(A \cup B)$ is a common lower bound of $s(A)$ and $s(B)$ in the specialization order of \mathbb{T}^ω . But as $s(A)$ and $s(B)$ differ in infinitely many positions, they cannot have a common lower bound in $\{p \in \mathbb{T}^\omega \mid |\mathbb{N} \setminus \text{dom}(p)| < \infty\}$. \square

Corollary 8. In Corollary 5, we cannot demand $s_{\{0,1\}^\omega}$ to be singlevalued.

We can now prove the restricted case of our main theorem for Cantor space. Note that $t_{\{0,1\}^\omega}$ almost but not quite satisfies the criteria, as names for singletons still contain a \perp .

Theorem 9. There are matching representations of $\{0,1\}^\omega$ and $\mathcal{K}(\{0,1\}^\omega)$.

Proof. We modify $t_{\{0,1\}^\omega}$ from Corollary 5 to yield a suitable representation ψ of $\mathcal{K}(\{0,1\}^\omega) \cong \mathcal{A}(\{0,1\}^\omega)$. Let $\psi(\langle p, q \rangle) = t_{\{0,1\}^\omega}(\perp q)$ if $\forall n \in \mathbb{N} p(n) \neq 1$ and $\psi(\langle p, q \rangle) = \emptyset$ if $\exists n \in \mathbb{N} p(n) = 1$. By taking into account the definition of $\delta_{\mathbb{T}}$ and Corollary 5, this is indeed a representation of $\mathcal{A}(\{0,1\}^\omega)$ (i.e. induces the correct computability notion of $\{0,1\}^\omega$).

The corresponding representation δ of $\{0,1\}^\omega$ is defined as follows: $\delta(\langle 0^\omega, q \rangle) = p$ iff $\forall n \in \mathbb{N} q(\nu^{-1}(p_{\leq n})) = p(n+1)$. It is easy to verify that $(\{0,1\}^\omega, \text{id}) \cong (\{0,1\}^\omega, \delta)$. \square

4 Proper dyadic subbases

In order to obtain a result akin to Theorem 9 for a large class of spaces, we will utilize the notion of a proper dyadic subbase. These were introduced in [11], and further studied in [6, 14, 13, 12]. The original motivation was to generalize the role of the binary and signed binary representations of real number: A proper dyadic subbase induces both (1) a “tiling” coding generalizes the binary expansion and (2) “covering” coding (which forms an admissible representation) generalizing the signed binary expansion.

The definition of a (not necessarily proper) dyadic subbase was changed in [13] compared to the previous literature. Here, we adopt the definition from [13].

Definition 10. A *dyadic subbase* of a represented space \mathbf{X} is a map $S : \mathbb{N} \times \{0,1\} \rightarrow \mathcal{O}(\mathbf{X})$ such that the image is a subbase of \mathbf{X} and $S(n,0) \cap S(n,1) = \emptyset$ for every $n \in \mathbb{N}$.

We write $S_{n,i}$ for $S(n,i)$ and $S_{n,\perp} = X \setminus (S_{n,0} \cup S_{n,1})$. A dyadic subbase S defines a (continuous) map φ_S from \mathbf{X} to \mathbb{T}^ω as follows.

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \perp & (x \in S_{n,\perp}). \end{cases}$$

That the image of S is a subbase ensures that φ_S even is a topological embedding. We call a dyadic subbase computable, if the map $S : \mathbb{N} \times \{0,1\} \rightarrow \mathcal{O}(\mathbf{X})$ is computable (which implies φ_S to be computable), and moreover, φ_S^{-1} is computable, too. This corresponds to the basis induced by the subbasis to be an effective countable basis in the sense of [2, Definition 11].

For a dyadic subbase S and $p \in \mathbb{T}^\omega$, define

$$S(p) = \bigcap_{k \in \text{dom}(p)} S_{k,p(k)}, \tag{1}$$

$$\bar{S}(p) = \bigcap_{k \in \text{dom}(p)} (X \setminus S_{k,1-p(k)}) = \bigcap_{k \in \text{dom}(p)} (S_{k,p(k)} \cup S_{k,\perp}). \tag{2}$$

$\{S(e) \mid e \in \mathbb{T}^*\}$ is the base of X generated by the subbase S . We have

$$x \in S(p) \Leftrightarrow \varphi_S(x)(k) = p(k) \text{ for } k \in \text{dom}(p) \Leftrightarrow \varphi_S(x) \succeq p, \tag{3}$$

$$x \in \bar{S}(p) \Leftrightarrow \varphi_S(x)(k) \sqsubseteq p(k) \text{ for } k \in \text{dom}(p) \Leftrightarrow \varphi_S(x) \uparrow p. \tag{4}$$

These equations show that S and \bar{S} are order-theoretic notion in \mathbb{T}^ω .

Definition 11. We say that a dyadic subbase S is *proper* if $\text{cl } S(e) = \bar{S}(e)$ for every $e \in \mathbb{T}^*$.

As a special case, if S is a proper dyadic subbase, $S_{n,0}$ and $S_{n,1}$ are regular open sets which are exteriors of each other. That is, $S_{n,\perp}$ is the common boundary between them and $\text{cl } S_{n,i} = S_{n,i} \cup S_{n,\perp}$.

The equivalences (3) and (4) show that a sequence $\varphi_S(x)$ not only contains information on the basic open sets x belongs to, but also information on the basic open sets to whose closure x belongs.

Proposition 12. Let $(S_{n,i})_{(n,i) \in \mathbb{N} \times \{0,1\}}$ be a computable dyadic subbase of \mathbf{X} . Then $p \mapsto S(p) : \mathbf{3}^* \rightarrow \mathcal{O}(\mathbf{X})$ and $p \mapsto \bar{S}(p) : \mathbb{T}^\omega \rightarrow \mathcal{A}(\mathbf{X})$ are computable.

Proof. This follows immediately from the definition of $S(p)$ and $\bar{S}(p)$ together with the observation that finite intersection $\cap : (\mathcal{O}(\mathbf{X})^* \rightarrow \mathcal{O}(\mathbf{X}))$, countable intersection $\bigcap : \mathcal{A}(\mathbf{X})^\omega \rightarrow \mathcal{A}(\mathbf{X})$ and complement $^C : \mathcal{O}(\mathbf{X}) \rightarrow \mathcal{A}(\mathbf{X})$ are computable (eg [7]). \square

Corollary 13. Let $(S_{n,i})_{(n,i) \in \mathbb{N} \times \{0,1\}}$ be a proper computable dyadic subbase of \mathbf{X} . Then $p \mapsto \bar{S}(p) : \mathbf{3}^* \rightarrow (\mathcal{A}(\mathbf{X}) \wedge \mathcal{V}(\mathbf{X}))$ is computable.

Proof. As $w \mapsto w \perp^\omega : \mathbf{3}^* \rightarrow \mathbb{T}^\omega$ is computable, we obtain $p \mapsto \bar{S}(p) : \mathbb{T}^* \rightarrow \mathcal{A}(\mathbf{X})$ from Proposition 12. By definition of proper dyadic subbase, $\bar{S}(p) = \text{cl } S(p)$ for $p \in \mathbb{T}^*$, and $\text{cl} : \mathcal{O}(\mathbf{X}) \rightarrow \mathcal{V}(\mathbf{X})$ is computable (eg [7]). We thus obtain $p \mapsto \bar{S}(p) : \mathbb{T}^* \rightarrow \mathcal{V}(\mathbf{X})$. \square

Definition 14. For a dyadic subbase S of a space \mathbf{X} , $p \in \mathbb{T}^\omega$ and $n \in \mathbb{N}$, we define $S_{\text{ex}}^n(p) \subseteq X$ and $\bar{S}_{\text{ex}}^n(p) \subseteq X$ as follows.

$$S_{\text{ex}}^n(p) = \bigcap_{k < n} S_{k,p(k)},$$

$$\bar{S}_{\text{ex}}^n(p) = \bigcap_{k < n} \text{cl } S_{k,p(k)}.$$

Note that $\text{cl } S_{n,\perp} = S_{n,\perp}$. For $e \in \mathbb{T}^*$, we define $S_{\text{ex}}(e) = S_{\text{ex}}^{|e|}(e)$ and $\bar{S}_{\text{ex}}(e) = \bar{S}_{\text{ex}}^{|e|}(e)$. We have

$$x \in S_{\text{ex}}^n(e) \Leftrightarrow \varphi_S(x)|_{\leq n} = e, \quad (5)$$

$$x \in \bar{S}_{\text{ex}}^n(e) \Leftrightarrow \varphi_S(x)|_{\leq n} \preceq e. \quad (6)$$

The sets $S_{\text{ex}}^n(p)$ may fail to be either open or closed, but are guaranteed to be Δ_2^0 . While each $\bar{S}_{\text{ex}}^n(p)$ is a closed set, this does not yield an effective statement in general: The sets $\text{cl } S_{k,0}$, $\text{cl } S_{k,1}$ will be available as elements of $\mathcal{V}(\mathbf{X})$, whereas $S_{n,\perp}$ is available as an element of $\mathcal{A}(\mathbf{X})$. Moreover, finite intersection is not a continuous operation on $\mathcal{V}(\mathbf{X})$. However, if S is proper, then $\text{cl } S_{k,0} = S_{k,1}^C$ and $\text{cl } S_{k,1} = S_{k,0}^C$ – i.e. all component sets of $\bar{S}_{\text{ex}}^n(p)$ are available as elements of $\mathcal{A}(\mathbf{X})$, and this space is effectively closed under intersection. We thus find:

Observation 15. Let S be a proper computable dyadic subbase of \mathbf{X} . Then $\bar{S}_{\text{ex}} : \mathbb{N} \times \mathbf{3}^\omega \rightarrow \mathcal{A}(\mathbf{X})$ is computable.

Definition 16. Let S be a dyadic subbase of a space \mathbf{X} . We define the poset $\hat{K}_S \subseteq \mathbb{T}^*$ as:

$$\hat{K}_S = \{p|_m \mid \exists x \in X \varphi_S(x) \preceq p, m \in \mathbb{N}\} = \{e \in \mathbb{T}^* \mid \bar{S}_{\text{ex}}(e) \neq \emptyset\}$$

We say that $e \in \mathbb{T}^*$ is an immediate successor of $d \in \mathbb{T}^*$ if $d \prec e$ and there is no element f such that $d \prec f \prec e$. A subset $L \subseteq \mathbb{T}^*$ is called finitely branching if each $d \in L$ has only finitely many immediate successors $e \in L$.

Theorem 17. (Proposition 5.10 and Theorem 6.5 of [14]) If S is a proper dyadic subbase of a compact Hausdorff space X , then \widehat{K}_S is a finitely branching poset.

Recall that we identify $e \in \mathbb{T}^*$ with $e \perp^\omega$ for the purpose of defining $\bar{S}_{\text{ex}}^n(e)$ for $n > |e|$. That \widehat{K}_S is finitely branching is equivalent to saying that for every e , there is n such that $\bar{S}_{\text{ex}}^n(e) = \emptyset$.

Lemma 18. Let S be a proper computable dyadic subbase of computably compact \mathbf{X} . There is a decidable set $H \subseteq \mathbf{3}^\omega$ and a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that⁽²⁾:

1. $\widehat{K}_S \subseteq H$
2. $H_{\langle n \rangle} \subseteq \mathbb{T}^{\leq g(n)}$

Proof. We define and compute the set H which contains \widehat{K}_S by inductively finding $H_{\langle n \rangle} \subset \mathbb{T}_{\langle n \rangle}^*$. Let $g(0) = 0$ and $H_{\langle 0 \rangle} = \{\epsilon\}$. Suppose that $H_{\langle n \rangle}$ is defined. For each element $d \in H_{\langle n \rangle}$, we perform the following procedure.

By dove-tailing, search for some $k_d \in \mathbb{N}$ such $\bar{S}_{\text{ex}}^{k_d}(d)$ is empty. By Theorem 17, \widehat{K}_S is finite-branching, hence such a k_d exists. We can recognize a suitable k_d by Observation 15 together with computable compactness of \mathbf{X} .

Then, set $g(n+1)$ to be $\max\{k_d \mid d \in H_{\langle n \rangle}\} - 1$ and $H_{\langle n+1 \rangle} = \{d \in \mathbb{T}_{\langle n+1 \rangle}^* \mid |d| \leq g(n+1)\}$. \square

5 Main result

We can now present and prove our main results. This will take the form of three theorems, each focusing on different aspects but sharing an underlying construction in the proof.

For strictly increasing $f : \mathbb{N} \rightarrow \mathbb{N}$, let $\mathbf{X}^{f(*)} := \bigcup_{n \in \mathbb{N}} \mathbf{X}^{f(n)}$. We call $T \subseteq \{0, 1\}^{f(*)}$ an f -tree, if it is closed under taking prefixes. For computable f , there is a straight-forward representation of the f -trees, and moreover, from an f -tree T we can compute a tree T' such that $T = T' \cap \{0, 1\}^{f(*)}$.

Theorem 19. Let \mathbf{X} be a computably compact Hausdorff space admitting a proper computable dyadic subbase. Then there is a computable strictly increasing $f : \mathbb{N} \rightarrow \mathbb{N}$ and a computable $R : \{0, 1\}^{f(*)} \rightarrow \mathcal{O}(\mathbf{X})$, $\bar{R} : \{0, 1\}^{f(*)} \rightarrow \mathcal{A}(\mathbf{X})$ with the following properties:

1. $R(\epsilon) = X$.
2. For $w \in \{0, 1\}^{f(n)}$, $R(w) = \bigcup_{v \in 2^{f(n+1)}, w \sqsubseteq v} R(v)$.
3. $\text{cl}R(w) = \bar{R}(w)$.
4. $(R(w))_{w \in \{0, 1\}^{f(*)}}$ is an effective countable basis of \mathbf{X} .
5. There is a computable $I : \{0, 1\}^{f(*)} \times \{0, 1\}^{f(*)} \rightarrow \{\emptyset\} \uplus \{0, 1\}^{f(*)}$ such that if $I(w, v) = \emptyset$, then $R(w) \cap R(v) = \emptyset$, and else $w \sqsubseteq I(w, v)$ and $R(w) \cap R(v) = R(I(w, v))$.
6. There is a computable $E : \{0, 1\}^{f(*)} \times \{0, 1\}^{f(*)} \rightarrow \{0, 1\}$ such that $E(w, v) = 1 \Rightarrow R(w) = R(v)$.

²Recall the convention that $\text{id} : \mathbf{3}^* \rightarrow \mathbb{T}^*$ is not written explicitly in order to make sense of the following statements.

$$7. \forall x \neq y \in \mathbf{X} \quad \exists n \in \mathbb{N} \quad \forall w, v \in \{0, 1\}^{f(n)} \quad (x, y \in \overline{R}(w) \wedge x, y \in \overline{R}(v) \Rightarrow E(w, v) = 1).$$

Proof. We start with the set H and the function g provided by Lemma 18. From them, we can easily obtain computable f and computable $r : \{0, 1\}^{f(*)} \rightarrow \mathbf{3}^*$ such that $r(\{0, 1\}^{f(n)}) = H_{\langle n \rangle}$, and $w \sqsubseteq v \Rightarrow r(w) \preceq r(v)$ and $r(w) \preceq x \Rightarrow \exists v \sqsupseteq w \ r(v) = x$. From r we then derive R and \overline{R} by setting $R(w) = S(r(w))$ and $\overline{R}(w) = \overline{S}(r(w))$.

Property (1) follows directly from the definition, Property (3) follows from the subbase being proper. Both (2) and (4) are a consequence of $\widehat{K}_S \subseteq H$.

For Property (5), note that if $S(w) \cap S(v) \neq \emptyset$, then $w \uparrow v$, and if $w \sqcup v$ denotes the least upper bound, then $S(w) \cap S(v) = S(w \sqcup v)$. The function I can be derived by following the construction of r , by ensuring that $r(I(w, v)) = r(w) \sqcap r(v)$ wherever the latter exists.

The function E from property (6) is defined by $E(w, v) = 1$ iff $r(w) = r(v)$. Property (7) then follows from S being a subbase of a Hausdorff space. \square

Theorem 20. Let \mathbf{X} be a computably compact Hausdorff space admitting a proper computable dyadic subbase. There is a surjection $t_{\mathbf{X}} : \mathbb{T}^\omega \rightarrow \mathcal{A}(\mathbf{X})$ and a multivalued map $s_{\mathbf{X}} : \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbb{T}^\omega$ such that

1. $s_{\mathbf{X}}$ and $t_{\mathbf{X}}$ are computable.
2. $t_{\mathbf{X}} \circ s_{\mathbf{X}} = \text{id}_{\mathcal{A}(\mathbf{X})}$.
3. If $A \in \mathcal{A}(\mathbf{X})$ is a finite set, then the cardinality of A is equal to the number of \perp in $s_{\mathbf{X}}(A)$.

Proof. We define the maps as having type $t_{\mathbf{X}} : \text{PT} \rightarrow \mathcal{A}(\mathbf{X})$ and $s_{\mathbf{X}} : \mathcal{A}(\mathbf{X}) \rightrightarrows \text{PT}$ instead, and then compose with δ_{PT} and δ_{PT}^{-1} . Our construction uses the maps f , I , E , R and \overline{R} from Theorem 19.

We define $t_{\mathbf{X}}(T) = \bigcap_{n \in \mathbb{N}} \bigcup_{w \in T \cap \{0, 1\}^{f(n)}} \overline{R}(w)$. As $\mathcal{A}(\mathbf{X})$ is effectively closed under finite unions and countable intersections, and since $w \notin T$ is recognizable, this does define a computable function.

Then $s_{\mathbf{X}}(A)$ is defined by a sequence T_0, T_1, \dots , where each $T_i \subseteq \{0, 1\}^{f(g(i))}$ for some computable function g such that $\forall w \in T_{i+1} \exists v \in T_i \ v \sqsubseteq w$. This sequence induces some tree $T \in \mathcal{T}$, and $s_{\mathbf{X}}(A)$ then is $\text{Prune}(T) \in \text{PT}$ following Theorem 4.

We start with $T_0 = \varepsilon$, and maintain throughout the process that $A \subseteq \bigcup_{w \in T_i} R(w)$. We search for (in a dove-tailing way) finite coverings $A \subseteq \bigcup_{w \in W} R(w)$. As \mathbf{X} is computably compact, we will find each such covering eventually - and there are infinitely many such coverings. If we do find a new covering W , then we consider $T_i \oplus W := \{I(v, w) \mid v \in T_i \wedge w \in W \wedge I(v, w) \neq \emptyset\}$. By definition of I , this ensures that $(\bigcup_{w \in T_i} R(w)) \cap (\bigcup_{w \in W} R(w)) = \bigcup_{w \in T_i \oplus W} R(w)$. We then choose $g(i+1)$ sufficiently big, and set $T_{i+1} := \{w \in 2^{f(g(i)+1)} \mid \exists v \in (T_i \oplus W) \ v \sqsubseteq w\}$ after removal of duplicates. With duplicates we mean $w \neq v$ such that $E(w, v) = 1$.

In addition, we search for $w \in T_j$ such that $\overline{R}(w) = \emptyset$, which we can identify due to the computable compactness of \mathbf{X} . If we find such an entry in stage i we set $T_{i+1} = \{v \in T_i \mid w \not\sqsubseteq v\}$ and $g(i+1) = g(i)$.

This procedure ensures that $\bigcap_{n \in \mathbb{N}} \bigcup_{w \in s_{\mathbf{X}}(A) \cap \{0, 1\}^{f(n)}} R(w)$ is equal to the intersection of all finite basic open coverings of A . In a compact Hausdorff space, this in turn implies $t_{\mathbf{X}}(s_{\mathbf{X}}(A)) = A$.

The remaining claim follows from the observation that if $|A| = k$, then any $T \in s_{\mathbf{X}}(A)$ will have k paths, together with the reasoning used in Corollary 5. This observation follows as

follows: Every infinite path through T induces a non-empty compact set (for if it were empty, already all but finitely many prefixes along the path would be mapped to the empty set, but these get removed). Two different paths cannot contain the same point by Property 4 in Theorem 19 together with the fact we are removing duplicate labels in the construction of T . \square

Theorem 21. Let \mathbf{X} be a computably admissible computably compact Hausdorff space admitting a proper computable dyadic subbase. There are matching representations of \mathbf{X} and $\mathcal{K}(\mathbf{X})$.

Proof. We can modify $t_{\mathbf{X}}$ from Theorem 20 in the same way as Theorem 9 was obtained from Corollary 5. \square

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